Infinite Series

Definitions:

1. Sequence: a list of numbers, in order, that follow a pattern
2. $f: N \to R$
    - A function that inputs natural numbers and maps them to real numbers
3. Series: the sum of the sequence

Examples of Infinite Series:

1. 1, 2, 3, … Natural Numbers
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ Zeno's Paradox
3. 1, 1, 2, 3, 5, 8, … Fibonacci Sequence
4. 1, -1, 1, -1, 1, … Alternating Yoyo (class name)
5. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ Harmonic Series
6. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$ Alternating Harmonic Series

Arithmetic Series:

$f(n) = f(n-1) + d$ Recursive

$f(n) = f(0) + dn$ Explicit

where d is the common difference

Geometric Series:

$f(n) = f(n-1) * r$ Recursive

$f(n) = f(o) * r^n$ Explicit

where r is the common ratio

Examples of Arithmetic and Geometric Series:

1. Explicit Arithmetic

$$f(n) = 5 + 2n$$

{5, 7, 9, …}

2. Explicit Geometric

$$f(n) = 5 * 2^n$$

{5, 10, 20, …}

Derivation of Geometric Series:

$$S_n = 1 + r + r^2 + \ldots + r^n$$
$$rS_n = r + r^2 + r^3 + \ldots + r^{n+1}$$
$$(1 - r)S_n = (1 + r + r^2 + \ldots + r^n) + (r + r^2 + r^3 + \ldots + r^{n+1})$$
$$(1 - r)S_n = 1 - r^{n+1}$$
$$S_n = \frac{1-r^{n+1}}{1-r} \text{ where } -1 < r < 1 \text{ as } n \to \infty$$

We know that $r^{n+1} \to 0$ as $n \to \infty$. Therefore,

$$S_n = \frac{1-0}{1-r}$$
$$S_n = \frac{1}{1-r}$$

Example of Geometric Series:

1. When $r = \frac{1}{3}$, the series looks like

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n}$$

Therefore,

$$S_n = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

2. When $r = \frac{2}{3}$, the series looks like

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots$$

Therefore,

$$S_n = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$$

The Comparison Test:

If you have two positive sequences a_n and b_n, and sequence a_n has terms smaller than b_n and Σb_n converges then Σa_n converges.

If you have two positive sequences a_n and b_n, and sequence b_n has terms larger than a_n and Σb_n diverges then Σa_n diverges.

Definition of an Alternating Series:

$(-1)^n * positive\ series$

Example of the Comparison Test:

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$
$$b_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

We know that Σb_n diverges and b_n has larger terms than a_n, therefore, Σa_n diverges.

Solving Series in Physics:

Physicians will tell you the following,

$$S = 1 + 2 + 3 + \ldots = -\frac{1}{12}$$

Here is the proof behind it. Given

$$S_1 = 1 - 1 + 1 - 1 + \ldots$$
$$S_2 = 1 - 2 + 3 - 4 + \ldots$$

We can add $S_1 + S_1$

$$S_1 + S_1 = 1 - 1 + 1 - 1 + \ldots$$
$$+ \quad 1 - 1 + 1 - \ldots$$
$$\overline{\quad 1 + 0 + 0 + 0 + \ldots \quad}$$

We can see that

$$S_1 + S_1 = 1 + 0 + 0 + 0 + \ldots$$
$$2\,S_1 = 1$$
$$S_1 = \frac{1}{2}$$

We can add $S_2 + S_2$

$$S_2 + S_2 = 1 - 2 + 3 - 4 + \ldots$$
$$+ \quad 1 - 2 + 3 - \ldots$$
$$\overline{\quad 1 - 1 + 1 - 1 + \ldots \quad}$$

We can see that

$$S_2 + S_2 = 1 - 1 + 1 - 1 + \ldots = S_1$$
$$2\,S_2 = S_1$$
$$S_2 = \frac{1}{4}$$

Now we can subtract $S - S_2$

$$S - S_2 = 1 + 2 + 3 + 4 + \ldots$$
$$-1 + 2 - 3 + 4 - \ldots$$
$$\overline{\quad 0 + 4 + 0 + 8 + \ldots \quad}$$

We can see that

$$S - S_2 = 4\,S$$

$$S - \frac{1}{4} = 4S$$

$$-\frac{1}{4} = 3S$$

$$S = -\frac{1}{12}$$